DGCI 2008

Binomial Convolutions and Derivatives Estimation from Noisy Discretization

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Addressed problem

The **estimation of derivatives** from possibly **noisy discrete functions** is important in frameworks such as image processing or shape analysis. This problem has been investigated through **finite difference methods** [1], **scale-space** [2], **discrete geometry** [3].

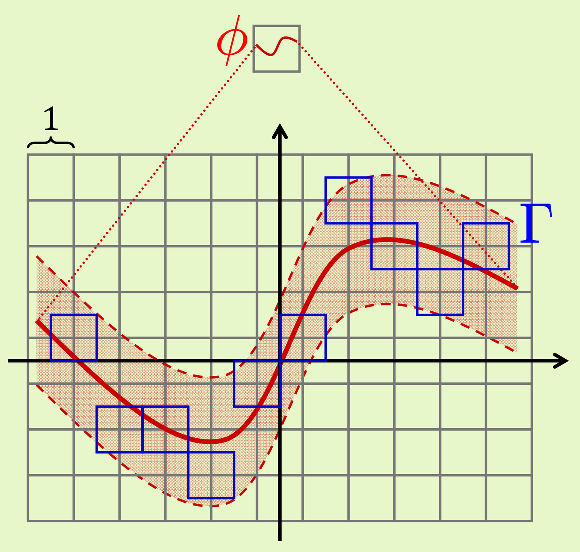
We present a **new approach** which relies on simple **discrete convolution products**. By means of the **binomial coefficients**, it uses **integer only** arithmetic (unlike scale-space). Besides, it has been proved to **converge** at rate $h^{2/3}$.

Noisy discrete function

$\Gamma : \mathbb{Z} \to \mathbb{Z}$

noisy discretization of

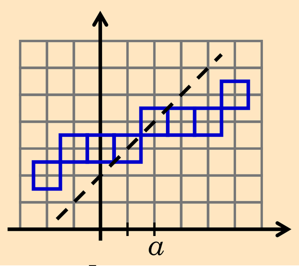
$\phi : \mathbb{R} \to \mathbb{R}$



$$|h\Gamma(a) - \phi(ha)| \leq Kh^{\alpha}$$

Finite differences kernel

$$\delta(a) = \begin{cases} 1 & \text{if } a = 0 \\ -1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$



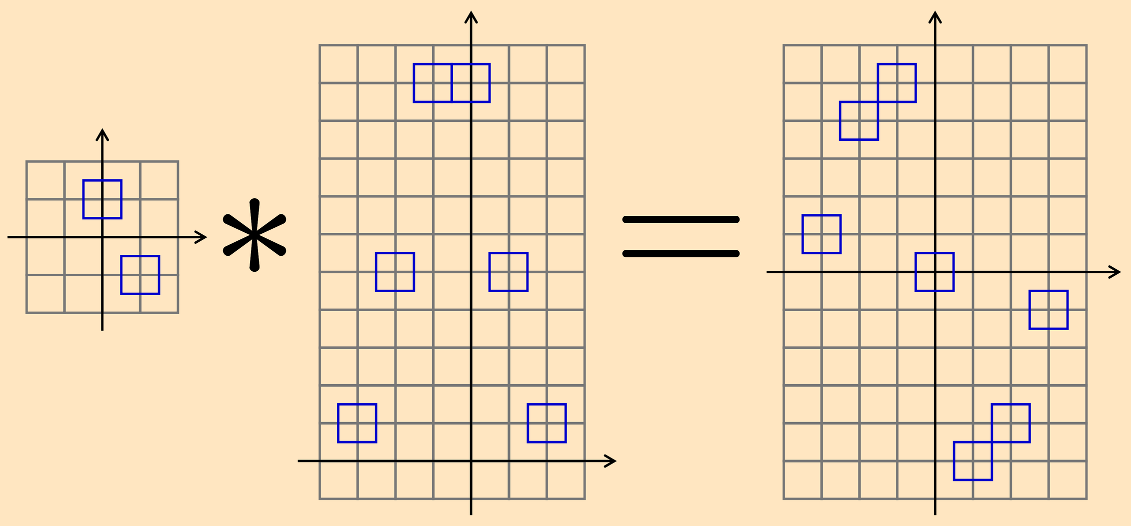
Smoothing kernel

$$H_n(a) = \begin{cases} \binom{n}{a+\frac{n}{2}} & \text{if } n \text{ is even and } a \in \{-\frac{n}{2}, \dots, \frac{n}{2}\} \\ \binom{n}{a+\frac{n+1}{2}} & \text{if } n \text{ is odd and } a \in \{-\frac{n+1}{2}, \dots, \frac{n+1}{2}\} \\ 0 & \text{otherwise} \end{cases}$$

	1						
	1	1					
	1	2	1				
	1	3	3	1			
0	**1**	**4**	**6**	**4**	**1**	0	
	1	5	10	10	5	1	

Derivative kernel

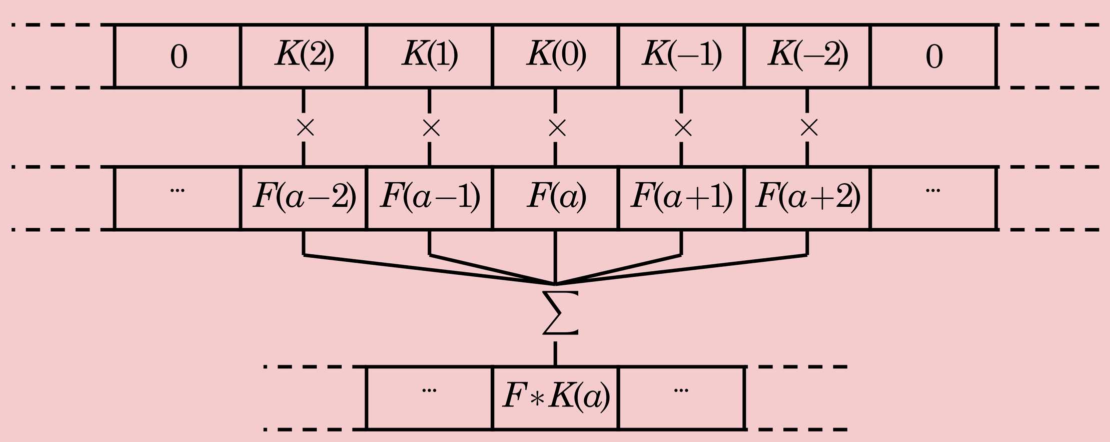
$$D_n(a) = \delta * H_n(a)$$



Discrete convolution product

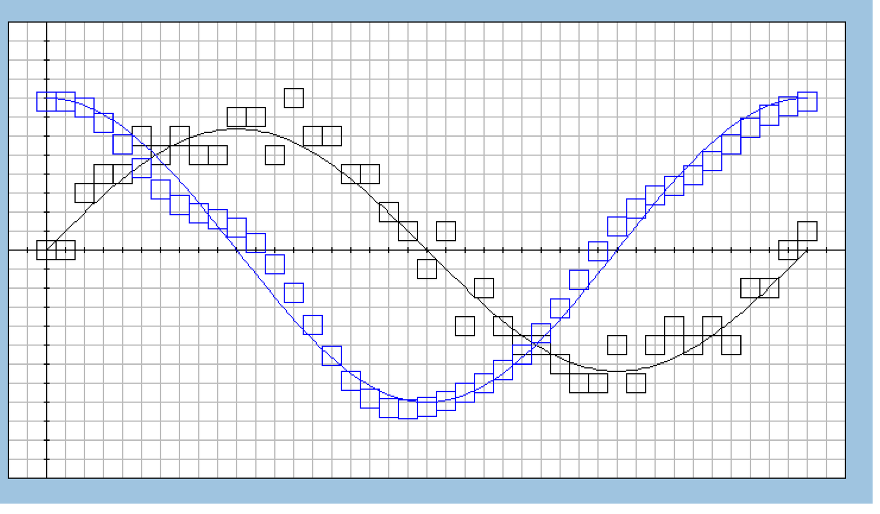
$F : \mathbb{Z} \to \mathbb{Z} \qquad K : \Omega \subset \mathbb{Z} \to \mathbb{Z}$

$$\Psi_K F(a) = K * F(a) = \sum_{i \in \Omega} K(i) F(a-i)$$



Derivative estimator

$$\phi'(ha) \approx \frac{1}{2^n} \Psi_{D_n} \Gamma(a)$$



Convergence

With the following hypothesis

- $\phi : \mathbb{R} \to \mathbb{R}$ is a C^3 function
- $\phi^{(3)}$ is bounded
- $\alpha \in]0, 1]$, $K \in \mathbb{R}_+^*$, and $h \in \mathbb{R}_+^*$
- $\Gamma : \mathbb{Z} \to \mathbb{Z}$ is such that: $|h\Gamma(a) - \phi(ha)| \leq Kh^{\alpha}$
- $n = \lfloor h^{2(\alpha-3)/3} \rfloor$

there exists a function $\sigma_{\phi,\alpha} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\sigma_{\phi,\alpha} \in O(h^{2\alpha/3})$ such that:

$$\left| \frac{1}{2^n} \Psi_{D_n} \Gamma(a) - \phi'(ha) \right| \leq \sigma_{\phi,\alpha}(h)$$

In other words, our derivative estimator converges at rate $h^{2\alpha/3}$ for functions known through their discretizations with step h and an arbitrary noise bounded by Kh^{α}.

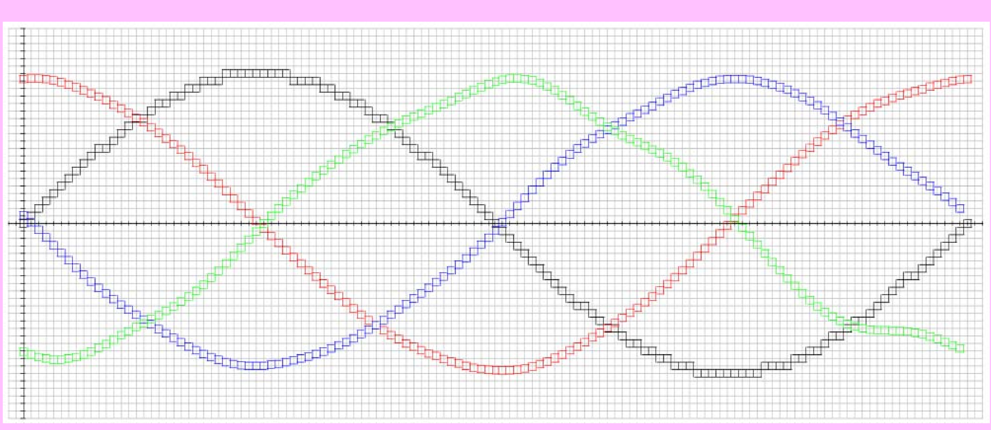
This result of convergence is proved using the Floater's theorem [4] (which gives the convergence of Bernstein approximations).

Higher order derivatives

Our estimator can be extended to the kth derivation order by using this kernel:

$$D_n^k = \underbrace{\delta * \ldots * \delta}_{k \text{ times}} * D_n$$

This estimator has also been proved to converge (by iterating the convergence result for first order derivatives).



Parametric functions

Pixel-length parametrization

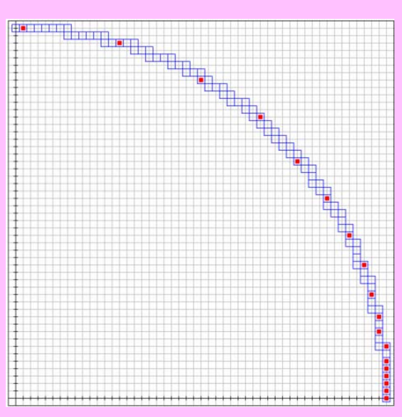
$\phi : \mathbb{R} \to \mathbb{R}^2$

$\Gamma : \mathbb{Z} \to \mathbb{Z}^2$

$$L(u) = \int_0^u \|\phi'(t)\|_1 \, dt$$

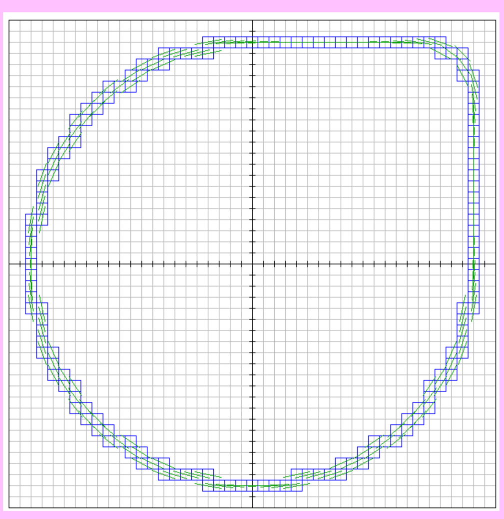
$$\overline{\phi}(x) = \phi(L^{-1}(x))$$

$$\Rightarrow \|h\Gamma(a) - \phi(ha)\| \leq Kh^{\alpha}$$



Tangent estimation

Estimation of the derivatives in the parametric case seems to work experimentally. However, there are some difficulties in proving the convergence: the pixel-length reparametrization is not C^3.



References

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